Surface Excitations of a Compressible Cylindrical Liquid

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The stability of an infinitely long, inviscid but compressible liquid jet under sinusoidal disturbances of the surface, still assumed to be sharp, is considered. This extends Rayleigh's theory. We show that despite compressibility the unstable region is $\lambda > \lambda_c = 2\pi a$ (λ wavelength, a radius) as Rayleigh's criterion for incompressible liquids states. The character of the unstable long wavelength mode changes with compressibility showing a critical onset if the inverse compressibility is only half the surface pressure. The maximum decay rate increases with compressibility and the most unstable wavelength increases too. An infinity of sound-like modes exists; their dispersion relations and flow fields are also given.

1. Introduction

In his pioneering work [1] Lord Rayleigh studied the stability of a liquid, inviscid, incompressible jet moving with uniform velocity (that can be taken as zero by transforming to the moving frame) under surface disturbances. His famous result was: the cylindrical file is unstable whenever the wavelength λ of the disturbance exceeds a critical value λ_c,

$$\lambda > \lambda_c = 2\pi a, \qquad (1)$$

and is stable under short wavelength disturbances. a is the radius of the undisturbed jet.

This simple geometric criterion, independent of surface tension α, proves to be rather robust if the properties of the system are made more realistic. For instance, it is not changed at all if finite viscosity is admitted, as shown in [2] for azimuthal symmetric modes and in [3] for all modes. It is hardly changed if the jet has finite length $2l$ instead of being infinitely long, see [4], [5]. There are only second order corrections, if a small thickness d of the surface is taken care of [6], $\propto (\mathrm{d}/a)^2$. If the liquid is uniformly charged, still (1) is approximately valid unless the jet's fissility $f = \varrho_E^2 a^3 / 2\alpha\varepsilon_0$ becomes large (f being the ratio of the Coulomb energy to twice the surface energy; ϱ_E charge density, ε_0 dielectric constant), as was derived in [3].

In the present paper we allow for another realistic feature of the liquid, namely for a finite compress-

ibility $\varkappa_0 = \varrho_0^{-1} \partial\varrho_0/\partial p_0$ (adiabatic). ϱ_0 is the equilibrium density and $\mathrm{d}p = (\partial p/\partial\varrho)_0 \,\mathrm{d}\varrho$ is the equation of state, taken at the equilibrium pressure $p_0 = \alpha/a$ due to surface tension. And still we shall find as a main result that the geometrical Rayleigh criterion (1) governs the jet's stability completely unchanged. Our basic assumption is that the surface is still sharp, which allows to solve the linearized fluid equations with the well-known boundary conditions exactly.

This is surprising since (1) usually is understood by a simple energy argument: There is a potential energy change due to a surface deformation described by a change

$$\zeta(t, \varphi, z) = \varepsilon_{00}(t) + \varepsilon(t)\cos(kz)\cos(s\varphi) \qquad (2)$$

of the jet's local radius $1 + \zeta$ along the jet's z-axis, where here and henceforth all lengths are measured in multiples of the undisturbed radius a; the wavenumber $k = (2\pi\lambda^{-1})a$ is dimensionless as is the azimuthal node number s. $\varepsilon(t)$ (short hand for $\varepsilon_{ks}(t)$) is the amplitude of the disturbance, ε_{00} a shift of the mean radius. Of course, $\varepsilon_{00} = 0$ if $\varepsilon = 0$, but ε_{00} necessarily is nonzero for a finite disturbance $\varepsilon \neq 0$. For incompressible fluids this is a consequence of volume conservation: If $\varepsilon_{00} = 0$ the positive half waves $(\cos(kz) > 0)$ contain more volume then the negative ones $(\cos(kz) < 0)$ can provide. Therefore the mean radius must shrink,

$$\varepsilon_{00} = -\varepsilon^2 (1 + \delta_{0s})/8. \qquad (3)$$

Now, the two contributions to the change in potential energy are (i) an increase $\propto k^2$ due to the

surface wigglyness and (ii) a decrease, independent of k, due to the mean radius reduction. If $k < k_c = 1$ the latter wins. This yields (1).

For compressible fluids, instead of volume conservation one argues with mass conservation, of course. But in addition to surface energy there is also compressional, i.e. internal energy which one expects to alter the onset of instability (1). Surprisingly enough the various contributions of compressional energy from surface change, namely in the internal energy *and* in the kinetic energy completely cancel. We shall derive the following expression for the total energy per length, sum of kinetic and potential energy T and U,

$$E = T + U = \frac{1 + \delta_{0s}}{4} \left\{ \frac{q_s(K)}{2} \dot{\varepsilon}^2 + \frac{k^2 + s^2 - 1}{2} \varepsilon^2 \right\}. \quad (4)$$

E is measured in multiples of $4\pi\alpha l/a$, the surface energy of the equilibrium state. The stiffness $k^2 + s^2 - 1$ does not depend on the compressibility. It is always positive if $s \geq 1$. Instability therefore occurs only if $s = 0$ and $k < 1$, implying (1). Compressibility only affects the inertia parameter q_s via its argument

$$K := \sqrt{k^2 + C\sigma^2}. \quad (5)$$

σ is the eigenvalue, real for the unstable mode, purely imaginary, $\sigma \equiv i\omega$, for the stable mode.

$$C = \varkappa_0 p_0 = \varkappa_0 \alpha/a \quad (6)$$

is the dimensionless compressibility. It is $C \approx 10^{-9}$ for water and $C \approx 0.27/(a/\mathrm{fm}) \approx 0.09$ for nuclear matter with radius $a = 3$ fm (if Blaizot's [7] value for $\varkappa_0$ is taken).

The functional form of the mass is the same as for incompressible fluids [3],

$$q_s(K) = \begin{cases} I_s(K)/K I_s'(K), & K \text{ real}, \\ J_s(|K|)/|K| J_s'(|K|), & K \text{ imaginary}. \end{cases} \quad (7)$$

Note that T as well as U both comprise compressional energy contributions individually. These are not visible in (4) since they cancel in the sum. Effectively (4) is a harmonic oscillator with frequency $\omega = \sqrt{(k^2 + s^2 - 1)/q_s(K)}$ or decay rate σ for negative stiffness, i.e. the eigenvalue equation (dispersion relation) reads

$$\sigma^2 q_s(\sqrt{k^2 + C\sigma^2}) = 1 - k^2 - s^2, \quad (8)$$

our main result.

Although this can be derived without calculating the mean radius shift ε_{00} one can determine this quantity also.

$$\varepsilon_{00}(t) = -\varepsilon^2(t) \frac{1 + \delta_{0s}}{8} [1 - 2C(1 - s^2 - k^2)]. \quad (9)$$

Note that only in the $s = 0$ long wavelength limit $k \to 0$ compressibility tends to decrease the mean radius reduction; there may even happen a blow up of the file if $C > [2(1 - s^2 - k^2)]^{-1}$. In the stable regime, however, $s^2 + k^2 > 1$, the mean radius of a compressible jet under surface disturbance is even smaller than for an incompressible one.

There is a seemingly small but in fact most remarkable effect of nonzero compressibility $C \neq 0$ in the dispersion relation (8). If $C = 0$ there are precisely two solutions $\pm\sigma$ or $\pm\omega$ of (8). If, instead, $C \neq 0$ the dispersion relation (8) has in addition to these basic modes (which are slightly changed) an infinity of further solutions, all representing sound-like oscillations in space and time. Basically $\omega \approx C^{-1/2} k$, identifying the sound velocity as $c = 1/\sqrt{C}$. The infinity of such modes can be labelled by their node number μ in radial direction, the difference between adjacent mode's frequencies being $\approx \pi/\sqrt{C}$. Since real physical systems are never absolutely incompressible these sound-like modes are always present, but in the unstable range they are by a factor $1/\sqrt{C}$ well above the basic mode. For large k the basic mode $\omega(k)$ turns out to approach $\omega = ck$ from below, i.e. the effective velocity is $\leq c$, subsonic, while all the other modes are supersonic, i.e. approach $\omega = ck$ from above.

Having thus summarized and discussed our results we shall now derive them in Sect. 2 and offer details and figures of the eigenspectrum and eigenfunctions in Section 3.

2. Linear Stability Analysis

The basic equations are the continuity equation (mass conservation) and Euler's equation (momentum conservation). No thermal effects are included.

a) Eigenfunctions

The linear deviations from equilibrium satisfy the equations of motion

$$\partial_t \varrho + \partial_i v_i = 0, \quad \text{continuity equation}, \quad (10)$$

$$\partial_t v_i + \partial_i p = 0, \quad \text{Euler's equations}, \quad (11)$$

$$\varrho = Cp, \quad \text{equation of state}. \quad (12)$$

Here, ϱ, p, v_i are the density (in terms of the equilibrium density ϱ_0), the pressure (in terms of α/a), and the velocity components (in terms of $\sqrt{\alpha/a\varrho_0}$). The derivatives ∂_i are with respect to x_i (in terms of a) and ∂_t with respect to the time t (in terms of $a/\sqrt{\alpha/a\varrho_0}$). C was defined in (6).

The basic equations imply the wave equation

$$C\partial_t^2 p - \Delta p = 0 . \tag{13}$$

Since $\partial_t^2 p$ together with p is real, the temporal behaviour is either $\propto \exp(\sigma t)$ with real σ or $\propto \cos \omega t$, i.e. $\sigma = i\omega$, quite naturally, since no damping effect is present. Of course, σ is dimensionless too; the physical unit of the decay rate or frequency is $\sqrt{\alpha/\varrho_0 a^3}$, being $\approx 8 \text{ s}^{-1}$ $(a/\text{cm})^{-3/2}$ for water or $\approx 2\cdot10^{22}\text{ s}^{-1}$ $(a/\text{fm})^{-3/2}$ for nuclear matter.

With the ansatz $p \propto f(r) \cos(s\varphi) \cos(kz)$ one solves (13) in cylindrical coordinates r, φ, z and obtains

$$f'' + r^{-1}f' - (C\sigma^2 + k^2 + s^2 r^{-2})f = 0 . \tag{14}$$

This is the equation for the cylindrical Bessel functions, in particular for the modified functions I_s if the argument $K = \sqrt{k^2 + C\sigma^2}$ is real or for the usual ones J_s if K us purely imaginary, which happens if $\sigma = i\omega$ and $C\omega^2 > k^2$, i.e. for supersonic modes. (The other independent solution of (14) is excluded by regularity at $r = 0$.)

The amplitudes and frequencies of the eigenmodes are not determined from the equation of motion but by the accompanying boundary conditions. Here our assumption enters that even for compressible liquids the surface is sufficiently sharp. If so, the surface vector will be

$$\boldsymbol{R} = (1 + \zeta(t, \varphi, z), \varphi, z) \tag{15}$$

with ζ from (2), describing the change of the radius along the jet.

The boundary condition for the velocity reads in linearized form

$$\boldsymbol{v}\cdot\boldsymbol{e_r} = \dot{\boldsymbol{R}}\cdot\boldsymbol{e_r}, \quad \text{at} \quad r = 1 , \tag{16}$$

i.e. the flow velocity normal to the boundary equals the movement of the boundary itself. This determines the amplitude of the v-field, which by (11) yields the amplitude of the $\sigma^{-2}\partial_i\dot{p}$ field in terms of $\dot{\varepsilon}$. We find

$$p(r, \varphi, z; t) = -\varepsilon(t)\sigma^2(KI_s'(K))^{-1}$$
$$\cdot \cos(kz)\cos(s\varphi)I_s(Kr), \tag{17}$$

$$v(r, \varphi, z; t) = -\sigma^{-2}\operatorname{grad}\dot{p} , \tag{18}$$

and, of course, $\varrho = Cp$. Read J_s instead of I_s and take $|K|$ if K is imaginary. σ is still arbitrary.

There is a second boundary condition, namely, for the pressure at the assumed sharp boundary (see text books, e.g. [8], [9])

$$p = 2\varkappa, \quad \text{at} \quad r = 1 . \tag{19}$$

Here, $\varkappa$ denotes the linear deviation in the mean curvature of the deformed cylinder from its equilibrium value $1/2$. Up to a factor C density and pressure coincide, cf. (12), so (19) also represents the jump in the density from 0 to $1 + 2C\varkappa$ at the sharp boundary. Note that due to the linearization and the simple equation of state (12) the number of independent fields is not really enlarged by allowing for compression but only the form of the continuity equation has changed from div $\boldsymbol{v} = 0$ to (10). If, in addition, a sharp surface is assumed, it should not be astonishing that the dispersion relation originating from (19) has a similar form as in the incompressible case.

Remember first the mean curvature deviation if a cylindrical shape is slightly deformed, see e.g. [9] or [3].

$$2\varkappa = -(\partial_z^2 + 1 + \partial_\varphi^2)\,\zeta . \tag{20}$$

Inserting ζ from (2) (linear, i.e. $\varepsilon_{00} \cong 0$), using (19) and (17) at the boundary $r = 1$ immediately implies the basic dispersion relation (8) for the eigenvalues σ, real, or $i\omega$, imaginary, with q_s from (7).

b) Mass conservation

Although we have pointed out why the dispersion relation in the compressible case is expected to be so similar to that of incompressible liquids, one nevertheless might be surprised that there is no visible influence of the compressibility to the restoring force, i.e. the r.h.s. of (8). There should be (and are indeed) contributions from compressional energies. To get more physical insight let us study now the energetics of the disturbed jet.

Start with mass conservation which holds anyhow. The undisturbed liquid fills a volume $V_0 = \pi a^2 \cdot 2l$ and has a surface $A_0 \cong 2\pi a \cdot 2l = 2V_0/a$. In the disturbed state the volume and surface are different. Using (15), (2) for the disturbed surface we find the following relative changes:

$$\delta V/V_0 = 2\varepsilon_{00} + \varepsilon^2 \cdot (1 + \delta_{0s})/4 , \tag{21}$$

$$\delta A/A_0 = \varepsilon_{00} + \varepsilon^2 \cdot (1 + \delta_{0s})(k^2 + s^2)/8 . \tag{22}$$

While the derivation of (21) is clear, to find (22) use the formula (see e.g. [3])

$$A/A_0 = (a/4\pi l) \int\limits_{-l/a}^{+l/a} dz \int\limits_{0}^{2\pi} d\varphi (1 + \zeta) \sqrt{1 + \partial_z^2 \zeta + \partial_\varphi^2 \zeta}.$$

Here and later the limit $l \to \infty$ has to be taken, of course.

If the liquid is incompressible, $\delta V = 0$, one recovers ε_{00} according to (3) from (21). If not, ε_{00} is different; its value can be determined from mass conservation (after dividing by ϱ_0):

$$V_0 = \int\limits_V (1 + \varrho) \, dV = V_0 + \int\limits_{V_0} \varrho \, dV + \int\limits_{\delta V} dV + \int\limits_{\delta V} \varrho \, dV.$$

V_0 cancels, $\int\limits_{V_0} \varrho \, dV$ vanishes due to the z-oscillations, so mass conservation implies

$$\delta V + \int\limits_{\delta V} \varrho \, dV = 0. \tag{23}$$

In the incompressible case each term vanishes individually relating ε_{00} with ε^2. Now the sum of both vanishes, still relating ε_{00} with ε^2. Inserting the solution ϱ from (12), (17) the integral can be evaluated

$$\int\limits_{\delta V} \varrho \, dV = - V_0 C \varepsilon^2 \sigma^2 q_s(K) (1 + \delta_{0s})/2. \tag{24}$$

Equations (21) and (24) inserted in the mass conservation equation (23) yields

$$\varepsilon_{00} = - \varepsilon^2 (t) \frac{1 + \delta_{0s}}{8} [1 - 2 C \sigma^2 q_s(K)]. \tag{25}$$

If, in addition, the dispersion relation is used we find our already given result (9), which was discussed in the introduction. Note that ε_{00} depends on s and k nontrivially if $C \neq 0$.

With (25) inserted into (21) the volume change

$$\frac{\delta V}{V_0} = \frac{1 + \delta_{0s}}{2} (1 - k^2 - s^2) C \varepsilon^2 \tag{26}$$

is *positive* in the unstable range but *negative* for all stable modes; of course, it is $\propto C$ and $\propto \varepsilon^2$. So, if surface tension wins and the jet oscillates, its mean volume decreases, while it increases if the jet decays.

c) *Energetics*

We now evaluate the different relevant energies. This has to be done up to second order in ε, since

second order in energy is equivalent to first order in the equation of motion. This makes the analysis a bit more intricate.

Let us begin with the surface energy, which is the only one for incompressible liquids. It equals αA. If we choose the undisturbed surface energy αA_0 as unit the surface energy deviation by the disturbance reads

$$U_{surf} = \varepsilon_{00} + \varepsilon^2 (1 + \delta_{0s}) (k^2 + s^2)/8, \tag{27}$$

cf. (22). The *form* of U_{surf} is independent of the compressibility; it is only the value of ε_{00} that depends on C. If $C = 0$, the first term counterbalances the second one partially to imply Rayleigh's criterion and $\varepsilon_{00} < 0$.

Next, there is internal energy if the fluid is compressible. Since thermal effects are not considered, the internal energy per volume only depends on the density.

$$u (1 + \varrho) = u_0 + \varrho \left.\frac{\partial u}{\partial \varrho}\right|_0 + \frac{1}{2} \varrho^2 \left.\frac{\partial^2 u}{\partial \varrho^2}\right|_0.$$

From thermodynamics we have $p_{tot} = - (\partial U_{int, tot}/\partial V)_m$, so $(1 + p) p_0 = u + (1 + \varrho) \partial u/\partial \varrho$. We conclude

$$\left.\frac{\partial u}{\partial \varrho}\right|_0 = p_0 + u_0, \quad \left.\frac{\partial^2 u}{\partial \varrho^2}\right|_0 = p_0 \frac{\partial p}{\partial \varrho}\bigg|_0 = \frac{p_0}{C}.$$

Therefore the internal energy up to second order reads

$$u = u_0 + (p_0 + u_0) \varrho + \varrho^2 p_0/2 C. \tag{28}$$

For the change in the internal energy one now calculates

$$- p_0 \delta V + (p_0/2 C) \int\limits_{V_0} \varrho^2 \, dV,$$

where mass conservation (23) has been used. Besides the energy by volume change due to surface shift, $- p_0 \delta V$, there is a pure volume term $p_0 (C/2) \int\limits_{V_0} p^2 \, dV$. Let us measure the internal energy deviation in multiples of the undisturbed surface energy too, i.e. in units αA_0 or $2 \alpha V_0/a$.

$$U_{int} = - \frac{1}{2} \frac{\delta V}{V_0} + \frac{C}{4} \int\limits_{V_0} p^2 \frac{dV}{V_0}. \tag{29}$$

In view of (21) we observe that in the sum $U_{surf} + U_{int}$ the mean radius reduction ε_{00} cancels, independent of its actual value. The total potential

energy thus reads

$$U = \varepsilon^2 (k^2 + s^2 - 1)(1 + \delta_{0s})/8$$
$$+ (C/4 V_0) \int_{V_0} p^2 \, dV. \tag{30}$$

It consists of a surface contribution of an incompressible liquid and a compressional volume energy.

Finally the kinetic energy is determined from

$$T = \frac{1}{2} \int_{V_0} v^2 \frac{dV}{2 V_0}, \tag{31}$$

again measured in multiples of the undisturbed surface energy (remember $a A_0 = 2 V_0$). Using the potential representation (18), always real, we get

$$T = -\frac{\sigma^{-2}}{2} \oint_{A_0} \dot{p}\, v \cdot \frac{dA}{A_0} - \frac{\sigma^{-2} C}{4 V_0} \int_{V_0} \dot{p}^2 \, dV. \tag{32}$$

There is again a surface motion contribution which by the boundary condition (16) can be connected with $\dot{R}$ and thus $\dot{\varepsilon}$; together with $p \sim \varepsilon$ this yields the expected kinetic energy of the oscillator $\sim \dot{\varepsilon}^2$.

In addition we observe the compressional kinetic volume energy $\sim \dot{p}^2$. This contribution, surprisingly enough, does *not* enter the total energy! Namely, if σ is real, $\sigma^{-2} \dot{p}^2 = p^2$ and the second T-contribution precisely cancels the second U-contribution. And, if $\sigma = i \omega$, so $p \propto \cos \omega t$, the volume contributions add up as $p^2 - \sigma^2 \dot{p}^2 = p^2 + \omega^{-2} \dot{p}^2 \sim \cos^2 \omega t + \sin^2 \omega t = 1$ to give a time independent constant irrespective of the actual value of σ (or ω, respectively). This is why no compressional volume energy enters the total energy (4), obtained from (30) and (32), using (17), (18) to evaluate (32).

3. Eigenvalues and Eigenfunctions

We now solve the dispersion relation (8). Since for real σ the l.h.s. $\sigma^2 q_s(K)$ is positive, unstable modes only occur if $s = 0$ and $k < 1$. We therefore start with the eigensolutions for azimuthal symmetric disturbances.

a) Spectrum for $s = 0$

The dispersion relation reads

$$\sigma^2 q_0 (\sqrt{k^2 + C \sigma^2}) = 1 - k^2. \tag{33}$$

The effective mass $q_0(K) = I_0(K)/K I_0'(K)$ is positive for real K. The inertia decreases monotonously

with increasing K,

$$q_0(K) = \begin{cases} 2 K^{-2}(1 + K^2/8 + \ldots), & K \to 0, \\ K^{-1} + 0(K^{-2}), & K \to \infty. \end{cases} \tag{34}$$

These properties suffice to evaluate the dominant features of the unstable mode, $k < 1$. In the long wavelength limit $k \to 0$ either

$$\sigma = \pm k/\sqrt{2 - C}, \quad C < 2, \tag{35a}$$

i.e. the unstable mode is arbitrarily slow due to the infinitely increasing mass or, for $C > 2$, there is a finite decay rate

$$\sigma = \pm y(C)/\sqrt{C} = \begin{cases} \pm \sqrt{C}, & C \gg 2, \\ \pm 2\sqrt{(C-2)/C}, & C \gtrsim 2, \end{cases} \tag{35b}$$

where $y(C)$ is the unique, real solution of

$$y^2 q_0 (y) = C, \quad C \geq 2, \tag{36}$$

i.e., $y \cong C$ for large compressibility and $y = 2\sqrt{C - 2}$ if C near 2, inferred from (34).

The bifurcation-like behaviour of the long wavelength decay rate with increasing compressibility is displayed in Figure 1. $C_c = 2$ is a critical value, for which $\sigma(k)$ increases infinitely fast with k, i.e. $\partial \sigma(k; C)/\partial k$ at fixed C is analogous to a susceptibility.

Figure 2 provides the (numerically obtained) decay rates vs. k for various compressibilities C. Note that the most unstable mode is shifted to smaller k with increasing C and soon above $C_c = 2$ the homogeneous disturbance $k = 0$ is the one that decays fastest, Figure 3. We are not aware of substances with a very large $\varkappa_0 \alpha$ product; thus
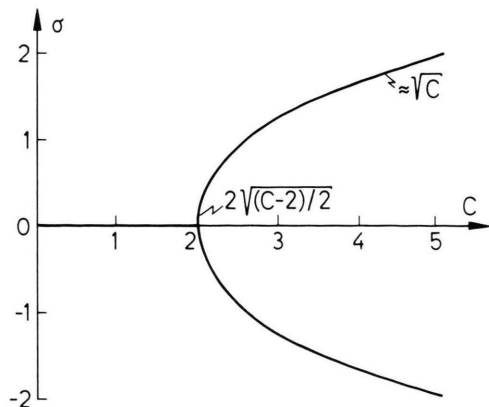

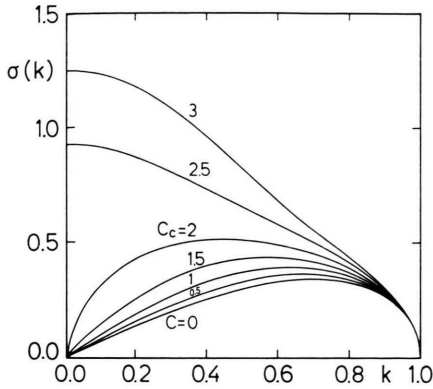Fig. 1. Decay rate $\sigma (k = 0)$ vs. compressibility C for $s = 0$.

Fig. 2. Decay rate $\sigma(k)$ vs. k in the unstable regime $0 \leq k \leq 1$ of the wavenumber in units of inverse radius a^{-1} and $s = 0$. The upper stability border $k_c = 1$ is dominated by the surface tension and is independent of compressibility; the long wavelength border is dominated by inertia, that decreases with compressibility C, the parameter for the various curves. (Here and in all later figures the branch $-\sigma$ is not shown.)
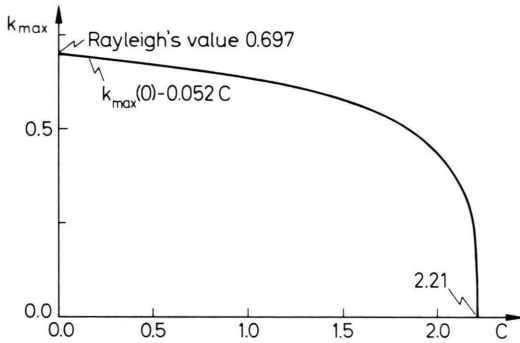

Fig. 3. Wavenumber k_{max} of the fastest decaying mode vs. compressibility C. If $C = 0$ this is Rayleigh's value $k_{max}(C = 0) = 0.697$.

$C = \varkappa_0 \alpha / a$ can be increased mainly via decrease of the jet radius a.

Besides the unstable mode there are other solutions of (33) which are purely imaginary $\sigma = i \omega$. ω solves

$$\omega^2 q_0 (\sqrt{k^2 - C \omega^2}) = k^2 - 1 \tag{37}$$

and $q_0(K)$ from (7). Consider first the long wavelength case $k \to 0$ again. If $C < 2$ there are *finite* oscillatory solutions (corresponding to the finite decay rates for $C > 2$) and if $C > 2$ the oscillator frequency is zero (corresponding in turn to the decay rate $\propto k$ for $C < 2$). The crossover again happens at $C_c = 2$. Of course, $k \approx 0$ means that q_0 is

given in terms of $J_0(\omega \sqrt{C})$, cf. (7). Due to the oscillatory character of this Bessel function there is an infinity of solutions

$$\omega_\mu = \pm y_\mu / \sqrt{C}, \quad k = 0, \tag{38}$$

separated approximately by $\pi / \sqrt{C}$ since adjacent zeros of J_0 satisfy $y_{\mu+1} - y_\mu \cong \pi$. Figure 4 shows the numerically obtained oscillator frequencies from (37) for $k = 0$. y_μ slightly depends on C.

If k is large there are no unstable modes. As long as C is small we can neglect it in (37), use the asymptotic behaviour (34) and find as the basic mode (well-known to Rayleigh already) $\omega \cong k^{3/2}$, starting from $\omega = 0$ at $k = 1$ proportional to $\sqrt{k-1}$. But, however small C might be, sooner or later $C \omega^2 \cong C k^3$ approaches k^2. At about $k_0 \cong 1/C$ (i.e. all the later the less compressible) this mode gets sound-like with a "refraction index" n

$$\omega = c \, k / n(k), \quad c := C^{-1/2} \text{ sound velocity},$$
$$n(k) \cong 1 + x^2 / 2 k^2, \quad \text{if} \quad k \gg x, \quad \text{with}$$
$$q_0(x) = C. \tag{39}$$

For small C the asymptotic range is reached if $k \gg 2/C$. Since $n > 1$ the effective velocity $\tilde{c} = c/n(k)$ is subsonic.

If C gets larger, the $k^{3/2}$ regime is shrinking, but there is always the subsonic basic mode, approaching $\omega = c \, k$ from below and starting at $k = 1$. For all C there are in addition infinitely many supersonic
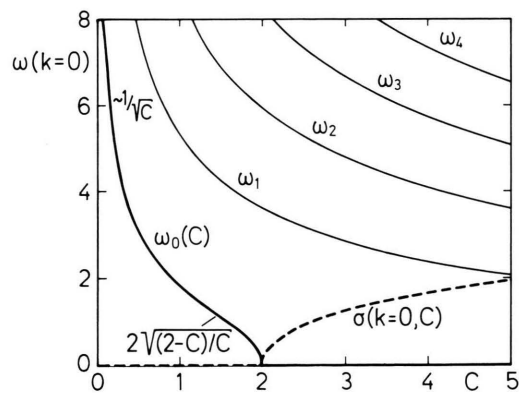

Fig. 4. Oscillator frequencies ω of long wavelength disturbances ($k = 0$) vs. compressibility C. The lowest (thick line) mode exchanges its behaviour with the unstable mode (dashed) at the critical compressibility $C_c = 2$. The bifurcation is opposite to that of the decay rate, cf. Fig. 1. If $C \to 0$ all these modes oscillate infinitely fast and cannot be observed.

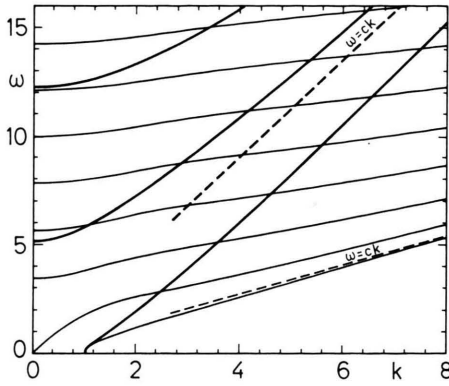Fig. 5. Oscillator frequencies ω vs. wavenumber k numerically obtained from the dispersion relation (37) for $s = 0$. The steeper curves (fat) belong to $C = 0.2$, the flatter ones to $C = 2.2$. The basic mode starts at $k_c = 1$ and is subsonic, all others are supersonic. The distance between two mode's frequencies is $\approx \pi/\sqrt{C}$. If $C > 2$ the lowest supersonic mode vanishes $\propto k$ in the long wavelength limit; for $C < 2$ it has a finite limit $\omega_0 (k \to 0)$, cf. Fig. 4. For sufficiently large k all modes approach sound, $\omega = c k$, broken lines. For fixed k the modes differ by their node number in the radial direction of the jet. If C is varied: the smaller C, the steeper the ω vs. k curves, the more they are separated, and the larger $\omega (k = 0)$. It is only above $C_c = 2$ that the lowest mode is soft, i.e. $\omega (k = 0) = 0$.

by μ corresponds to increasing "wavenumber" in radial direction. The basic mode always is different in character: it is a motion from the large radius parts of the jet to those with small radius, in its form rather independent of the compressibility. We show this in Figure 6.

For comparison we also offer a visualization of the flow field of an incompressible but viscous liquid under surface deformation in Figure 7. This was studied in [3]; in particular the explicit solution was given in (4.14). The basic mode again serves to bring the mass from the wave's maximum to its minimum and looks entirely similar to the non-viscous potential flow in Fig. 6, upper. The eigenfrequencies are real, since viscosity damps the flow. Stability is indicated by the minus sign. The mode with next larger σ (lower part Fig. 7) is entirely different: it clearly exhibits the vortex structure possible only and typical in viscous flows. Comparison of Figs. 6 and 7 elucidates the similarity of Rayleigh's basic modes independent of C, $\tilde{v}, \dots$. The differences of the systems show up in the modes that are not yet present if $C = \tilde{v} = 0$.

modes with

$$n(k) \cong 1 - x_\mu^2/2 k^2 , \qquad (40)$$

and x_μ solves $q_0(x_\mu) = C$ for large k. All sound-like modes are shown in Figure 5. In the case of quantized systems, these (at fixed C approximately equidistant) modes might be observable as elementary excitations, a signature of nonzero compressibility.

b) Eigenfunctions for $s = 0$

Knowing the eigenvalues, i.e. the decay rates σ or the frequencies ω, we can evaluate the form of the eigenfunctions, given in (17) and (18). Of course, the azimuthal node number is s, the node number per length along the z-axis is $k/2\pi$. The radial shapes of the flow and the pressure field (ϱ is C times pressure) are given by $I_0(Kr)$ or $J_0(|K|r)$ for unstable or oscillatory modes, respectively. If the eigenvalue ω corresponds to the smallest zero of $J_0(|K|)$ (viz. $q_0(|K|)$) there is no node in addition to $r = 0$ in the flow field; for the supersonic modes corresponding to the higher zeros there are more and more nodes if r increases from 0 to 1. Therefore the infinity of modes displayed in the earlier figures and labelled
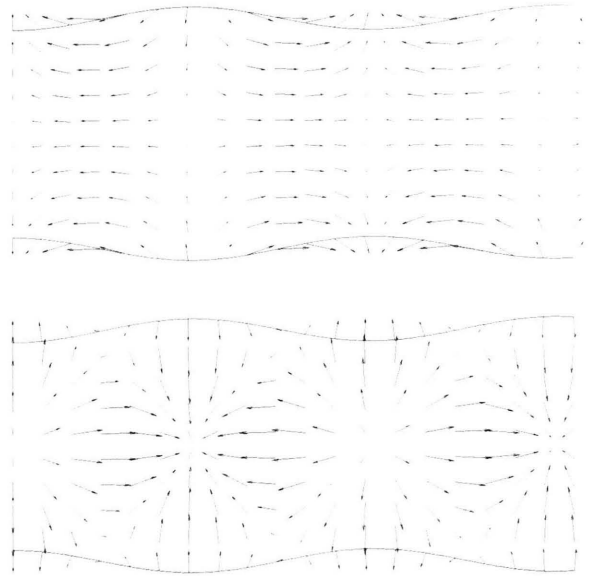

Fig. 6. Flow field for $k = 2$ (i.e. stable range), $s = 0$, $C = 0.5$. Upper: basic mode; lower: 1-node supersonic velocity field. The frequencies are $\omega \cong 1.71$ and $\omega \cong 4.78$, respectively. The mean radius is $1 + \varepsilon_{00}$, $\varepsilon_{00} = -0.0025$ (invisible). The arrows representing the velocity field are obtained by integrating the solution (18) for a short time.
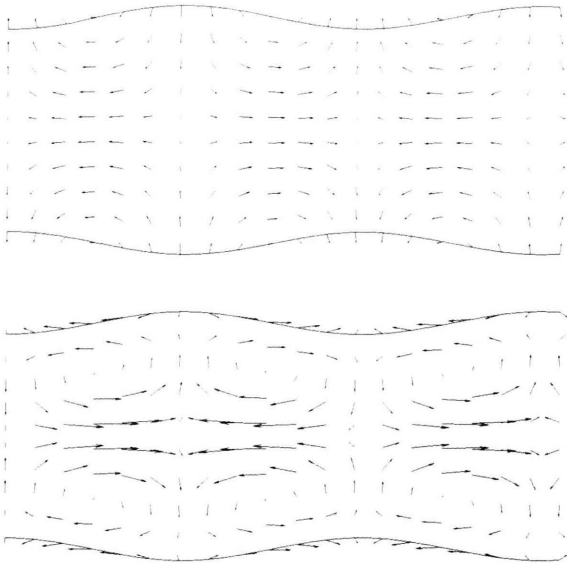
Fig. 7. Flow field of viscous but incompressible flow induced by a surface mode with $k = 2$, $s = 0$. $\tilde{v} = v/\sqrt{\alpha\, a/\varrho_0}$ $= 0.5$, with v the kinematic viscosity. Upper: basic mode, $\sigma = -3.26$; lower: first vortex mode, $\sigma = -10.13$.

c) Spectrum, all s

If $s \geqq 1$ the restoring surface potential $(s^2 + k^2 - 1)$ is always positive. Hence no mode with an azimuthal node, $s > 0$, can decay. All such surface disturbances oscillate. There is the well-known basic mode and in addition the supersonic sound modes. These have the eigenvalues $\sigma = \pm\, i\,\omega$,

$$\omega_\mu = c\, k/n_\mu(k)\,, \quad c = C^{-1/2} \text{ sound velocity}\,, \quad (41\,\text{a})$$

$$n_\mu(k) = 1/(1 + x_\mu^2 k^{-2})^{1/2}\,, \tag{41 b}$$

where x_μ solves

$$q_s(|x_\mu|) = C(k^2 + s^2 - 1)/(k^2 + x_\mu^2)\,,$$
$$q_s(|x|) \equiv J_s(|x|)/|x|\, J_s'(|x|)\,. \tag{41 c}$$

If k is sufficiently large this yields (40).

The $k \to 0$ behaviour for $s \neq 0$ is more regular than in the azimuthal symmetric case since the monotonously decreasing mass parameter never diverges,

$$q_s(K) = \begin{cases} 1/s\,, & K \to 0\,, \\ 1/K + 0\,(K^{-2})\,, & K \to \infty\,, \end{cases} \quad s \geqq 1\,. \tag{42}$$

Finite restoring force and finite inertia imply finite frequencies ω. But as the effective mass decreases with increasing s the corresponding ω gets larger with s.

Some approximate formulae might be useful. The ω_μ from (41) represent the exact dispersion relation (8). They can be evaluated approximately by approximating x_μ.

$$x_\mu \cong y_\mu(1 + \delta)\,, \quad \text{with} \quad J_s'(y_\mu) = 0\,, \tag{43}$$
$$\delta = (k^2 + y_\mu^2)\,(s^2 - y_\mu^2)\,[C\,(k^2 + s^2 - 1)\,(s^2 - y_\mu^2)^2$$
$$- 2 y_\mu^2(s^2 - y_\mu^2) - 0.5\,(s^2 + y_\mu^2)\,(k^2 + y_\mu^2)]^{-1}\,.$$

The zeros of J_s' are easily accessible, cf. for instance [10].

Another approximation allows to estimate the corrections due to compressibility provided C is still small and k is not too small. Define σ_0 as $\sigma(k, s, C = 0)$ for all k, s. Then for all k, s

$$\sigma = \sigma_0[1 + C\,\sigma_0^2(-q_s'(k)/4\,k\, q_s(k)) + 0\,(C^2)]\,. \tag{44}$$

Since q_s is monotonously decreasing, $-q_s' > 0$, thus the correction $\propto C$ has the sign of σ_0^2. Therefore unstable modes decay faster while oscillatory modes oscillate slower due to compressibility. This does not happen as a consequence of a change in stiffness but, instead, since compressibility decreases inertia for the unstable mode and increases inertia for the stable one, inferred from $q_s(\sqrt{k^2 + C\,\sigma^2})$.

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